On equivalence relations Σ^1_1-definable over $H(\kappa)$

Saharon Shelah[*]   Pauli Väisänen

November 12, 1999

Abstract

Let κ be an uncountable regular cardinal. Call an equivalence relation on functions from κ into 2 Σ^1_1-definable over $H(\kappa)$ if there is a first order sentence ϕ and a parameter $R \subseteq H(\kappa)$ such that functions $f, g \in {}^\kappa 2$ are equivalent iff for some $h \in {}^\kappa 2$, the structure $\langle H(\kappa), \in, R, f, g, h \rangle$ satisfies ϕ, where $\in$, R, f, g, and h are interpretations of the symbols appearing in ϕ. All the values μ, $1 \le \mu \le \kappa^+$ or $\mu = 2^\kappa$, are possible numbers of equivalence classes for such a Σ^1_1-equivalence relation. Additionally, the possibilities are closed under unions of $\le \kappa$-many cardinals and products of $< \kappa$-many cardinals. We prove that, consistent wise, these are the only restrictions under the singular cardinal hypothesis. The result is that the possible numbers of equivalence classes of Σ^1_1-equivalence relations might consistent wise be exactly those cardinals which are in a prearranged set, provided that the singular cardinal hypothesis holds and that the following necessary conditions are fulfilled: the prearranged set contains all the nonzero cardinals in $\kappa \cup \{\kappa, \kappa^+, 2^\kappa\}$ and it is closed under unions of $\le \kappa$-many cardinals and products of $< \kappa$-many cardinals. The result is applied in [SV] to get a complete solution of the problem of the possible numbers of strongly equivalent non-isomorphic models of weakly compact cardinality. [1]

---

1 Introduction

We deal with equivalence relations which are in a simple way definable over $H(\kappa)$ when κ is an uncountable regular cardinal. The conclusion will be that we can completely control the possible numbers of equivalence classes of such equivalence relations, provided that the singular cardinal hypothesis holds. The main application of this is the solution of the problem of the possible numbers of strongly equivalent non-isomorphic models of weakly compact cardinality. Namely, we prove in [SV] that when κ is a weakly compact cardinal, there exists a model of cardinality κ with μ-many strongly equivalent non-isomorphic models if, and only if, there exists an equivalence relation which is Σ_1^1-definable over $H(\kappa)$ and it has exactly μ different equivalence classes. The paper [SV] can be read independently of this paper, if the reader accepts the present conclusion on faith.

For every nonzero cardinals $\mu \leq \kappa$ or $\mu = 2^\kappa$, there is an equivalence relation Σ_1^1-definable over $H(\kappa)$ with μ different equivalence classes. There is also a Σ_1^1-equivalence relation with κ^+-many classes (Lemma 3.2). Furthermore, by a simple coding, the possible numbers of equivalence classes of Σ_1^1-equivalence relations are closed under unions of length $\leq \kappa$ and products of length $< \kappa$. In other words, assuming that $\gamma \leq \kappa$ and χ_i, $i < \gamma$, are cardinals such that for each $i < \gamma$, there is a Σ_1^1-equivalence relation having χ_i different equivalence classes, then there is a Σ_1^1-equivalence relation having $\bigcup_{i<\gamma} \chi_i$ different equivalence classes, and if $\gamma < \kappa$, there is also a Σ_1^1-equivalence relation with $\operatorname{card}(\prod_{i<\gamma} \chi_i)$ different equivalence classes (Lemma 3.4).

What are the possible numbers of equivalence classes between κ^+ and 2^κ? The existence of a tree $T \subseteq H(\kappa)$ with μ different κ-branches through it implies that there is a Σ_1^1-equivalence relation having exactly μ equivalence classes (Lemma 3.2). Therefore, existence of a Kurepa tree of height κ with more than κ^+-many and less than 2^κ-many κ-branches through it presents an example of a Σ_1^1-equivalence relation with many equivalence classes, but not the maximal number. On the other hand, in an ordinary Cohen extension of L, in which $2^\kappa > \kappa^+$, there is no definable equivalence relation having μ-many different equivalence classes when $\kappa^+ < \mu < 2^\kappa$ (a proof of this fact is straightforward, and in fact, it is involved in the proof presented in Section 4).

We show that, consistent wise, the closure properties mentioned are the only restrictions concerning the possible numbers of equivalence classes of Σ_1^1-

equivalence relations. Namely the conclusion will be the following: Suppose $\lambda > \kappa^+$ is a cardinal with $\lambda^\kappa = \lambda$ and Ω is a set of cardinals between κ^+ and λ so that it is closed under unions of $\leq \kappa$-many cardinals and products of $< \kappa$-many cardinals. We shall prove that after adding into L in the "standard" way Kurepa trees of height κ with μ-many κ-branches through it, for every $\mu \in \Omega$ (and repeating each addition λ-many times), there exists, in the generic extension, an equivalence relation Σ^1_1-definable over $H(\kappa)$ with μ-many equivalence classes if, and only if, μ is a nonzero cardinal $\leq \kappa^+$ or μ is in $\Omega \cup \{2^\kappa\}$.

In order to make this paper self contained, we introduce the standard way to add a Kurepa tree and give some basic facts concerning that forcing in Section 2. The essential points are the following. Firstly, if one adds several new Kurepa trees, the addition of new trees does not produce new κ-branches of the old trees. Secondly, permutations of "the labels" of the κ-branches of the generic Kurepa trees, determine many different automorphisms of the forcing itself. These kind of automorphisms can be used "to copy" two different equivalence classes of a definable equivalence relation to several different equivalence classes. In fact, this is "the straightforward way" to show that in an ordinary Cohen extension of L, a definable equivalence relation has either $\leq \kappa^+$-many, or the maximal number 2^κ-many equivalence classes. The main difference, however, between the standard "Cohen-case" and the proof presented in Section 4 is that the ordinary Δ-lemma cannot be directly applied as can be done in the former case.

In Section 3 we introduce proofs of some basic facts mentioned above. The crucial fact is that a Σ^1_1-equivalence relation is absolute for various generic extensions (Lemma 3.5 and Conclusion 3.6). The theorem is formally written in Section 4, and the proof of it is divided into several subsections. The main idea is the following. We start to look at an equivalence relation which is Σ^1_1-definable over $H(\kappa)$ using some parameter of cardinality κ. The forcing consist of addition of λ-many different trees. However, we may assume that the forcing name of the parameter has cardinality κ, and thus, there are only κ-many trees which really has "effect" on the number of classes of the fixed equivalence relation. So we restrict ourselves to the subforcing consisting of the addition of these κ-many "critical" trees. (Note, in Lemma 4.2 we introduce a subforcing consisting of addition of κ^+-many trees, but right after that in Subsection 4.2, we define "isomorphism classes" of names in order to concentrate only on κ-many generic trees.) Then as explained in Subsection 4.3, from our assumption that the singular cardinal hypothesis holds, it immediately follows that either 1) the fixed equivalence relation has

3

χ classes, where χ is a union of $\leq \kappa$-many cardinals or a product of $< \kappa$-many cardinals from the prearranged set Ω, or otherwise, 2) the number of equivalence classes really depends on κ-many trees, not less than κ-many. On the other hand, we know that the rest of the forcing, i.e., the addition of the other trees than those κ-many critical ones, produces λ-many new subsets of any set having size κ. So, when the equivalence depends on κ-many trees, we show in Subsection 4.4 that either 1) the fixed equivalence relation has χ classes, where χ is a union of κ-many products having length $< \kappa$ and cardinals from Ω, or otherwise, 2) the rest of the forcing produces λ-many new equivalence classes.

In Section 5 we give some concluding remarks.

2 Adding Kurepa trees

Throughout of this paper we assume that κ is an uncountable regular cardinal and $\kappa^{<\kappa} = \kappa$. For sets X and Y we denote the set of all functions from X into Y by ^{X}Y. For a cardinal μ, we let $[X]^{\mu}$ be the set of all subsets of X having cardinality μ.

The following forcing is the "standard" way to add a Kurepa tree [Jec71, Jec97].

Definition 2.1 *Let μ be a cardinal $\geq \kappa$. Define a forcing P_μ as follows. It consists of all pairs $p = \langle T^p, \langle b^p_\delta \mid \delta \in \Delta^p \rangle \rangle$ where*

> *for some $\alpha < \kappa$, T^p is a subset of $\{\eta \mid \eta \in {}^{\beta}2 \text{ and } \beta < \alpha\}$ such that it is of cardinality $< \kappa$ and closed under restriction;*

> *Δ^p is a subset of μ having cardinality $< \kappa$ and each b^p_δ is an α-branch trough T^p when T^p is ordered by the inclusion.*

For all $p, q \in P_\mu$, we define that $q \leq p$ if

> *T^q is an end-extension of T^p;*

> *$\Delta^p \subseteq \Delta^q$;*

> *for every $\delta \in \Delta^p$, b^q_δ is an extension of b^p_δ.*

Fact 2.2

a) P_μ is κ-closed and it satisfies κ^+-chain condition.

b) Suppose G is a P_μ-generic set over V. In $V[G]$, $T^G = \bigcup_{p \in G} T^p$ is a tree of height κ and each of its level has cardinality $< \kappa$.

Lemma 2.3 Let $\tilde{Q}$ be such that $\mathbf{1} \Vdash_{P_\mu}$ "$\tilde{Q}$ is a κ-closed forcing notion". Suppose G is a P_μ-generic set over V and H is Q-generic set over $V[G]$. Then, in $V[G][H]$, the κ-branches trough the tree $T^G = \bigcup_{p \in G} T^p$ are the functions b_δ^G, $\delta < \mu$, having domain κ and satisfying for every $\alpha < \kappa$ that $b_\delta^G(\alpha) = b_\delta^p(\alpha)$ for some $p \in G$ with $\delta \in \Delta^p$ and $\alpha \in \mathrm{dom}(b_\delta^p)$.

Proof. The idea of the proof is the same as in [Jec71]. Suppose $\langle p_0, \tilde{q}_0 \rangle$ is a condition in $P_\mu * \tilde{Q}$ and $\tilde{t}$ is a name such that

$$\langle p_0, \tilde{q}_0 \rangle \Vdash_{P_\mu * \tilde{Q}} \text{``} \tilde{t} \text{ is a } \kappa\text{-branch through } \tilde{T}^G \text{ and } \tilde{t} \notin \{\tilde{b}_\delta^G \mid \delta < \mu\}\text{''}.$$

Since $\mathbf{1} \Vdash_{P_\mu * \tilde{Q}}$ "κ is a regular cardinal", it follows that every condition below $\langle p_0, \tilde{q}_0 \rangle$ forces that for all $X \in [\mu]^{<\kappa}$ and $\beta < \kappa$, there is $\alpha > \beta$ with

$$\tilde{t}(\alpha) \notin \{\tilde{b}_\delta^G(\alpha) \mid \delta \in X\}.$$

Let α_0 be the height of T^{p_0}. Choose conditions $\langle p_n, \tilde{q}_n \rangle$ from $P_\mu * \tilde{Q}$ and ordinals α_n, $1 < n < \omega$, so that for every $n < \omega$, the height of the tree $T^{p_{n+1}}$ is greater than α_n, $\langle p_{n+1}, \tilde{q}_{n+1} \rangle \le \langle p_n, \tilde{q}_n \rangle$, and

(A) $\qquad \langle p_{n+1}, \tilde{q}_{n+1} \rangle \Vdash_{(P_\mu * \tilde{Q})} \tilde{t}(\alpha_{n+1}) \notin \{\tilde{b}_\delta^G(\alpha_{n+1}) \mid \delta \in \Delta^{p_n}\}.$

Define r to be the condition in P_μ satisfying $T^r = \bigcup_{n<\omega} T^{p_n}$, $\Delta^r = \bigcup_{n<\omega} \Delta^{p_n}$, and for every $\delta \in \Delta^r$, $b_\delta^r = \bigcup_{n \in (\omega \setminus m)} b_\delta^{p_n}$, where m is the smallest index with $\delta \in \Delta^{p_m}$. Then T^r is of height $\alpha = \bigcup_{n<\omega} \alpha_n$. In order to restrict the αth level of the generic tree, abbreviate the function $\bigcup_{\gamma < \alpha} b_\delta^r(\gamma)$, $\delta \in \Delta^r$, by f_δ, and define r' to be the condition in P_μ with $T^{r'} = T^r \cup \{f_\delta \mid \delta \in \Delta^r\}$, $\Delta^{r'} = \Delta^r$, and for every $\delta \in \Delta^{r'}$ and $\beta \le \alpha$,

$$b_\delta^{r'}(\beta) = \begin{cases} b_\delta^r(\beta) & \text{if } \beta < \alpha; \\ f_\delta & \text{if } \beta = \alpha. \end{cases}$$

Now r' forces that the αth level of the generic tree $\tilde{T}^G$ consist of the elements f_δ, $\delta \in \Delta^{r'}$.

Since r' forces $\tilde{Q}$ to be κ-closed and $\langle \tilde{q}_n \mid n < \omega \rangle$ to be a decreasing sequence of conditions, there is $\tilde{q}'$ so that $\langle r', q' \rangle \le \langle p_n, \tilde{q}_n \rangle$ for every $n < \omega$. Since

5

$\langle r', q' \rangle$ forces that $\tilde{t}(\alpha) \in \{f_\delta \mid \delta \in \Delta^{r'}\}$, there are $\delta \in \Delta^{r'}$ and a condition $\langle r'', \tilde{q}'' \rangle \leq \langle r', \tilde{q}' \rangle$ in $P_\mu * \tilde{Q}$ forcing that $\tilde{t}(\alpha) = f_\delta$. However, if n is the smallest index with $\delta \in \Delta^{p_n}$, then $\langle r'', \tilde{q}'' \rangle$ forces that

$$\tilde{t}(\alpha_{n+1}) = f_\delta \restriction \alpha_{n+1} = b_\delta^{r'}(\alpha_{n+1}) = b_\delta^{r''}(\alpha_{n+1}) = \tilde{b}_\delta^G(\alpha_{n+1}),$$

contrary to (A). $\blacksquare_{2.3}$

Definition 2.4 *Suppose $\lambda > \kappa^+$ is a cardinal with $\lambda^\kappa = \lambda$. Let $\bar{\mu} = \langle \mu_\xi \mid \xi < \lambda \rangle$ be a fixed sequence of cardinals such that $\kappa < \mu_\xi \leq \lambda$ and for every $\chi \in \{\mu_\xi \mid \xi < \lambda\} \cup \{\lambda\}$, the set $\{\zeta < \lambda \mid \mu_\zeta = \chi\}$ has cardinality λ. We define $\mathbb{P}(\bar{\mu})$ to be the product of P_{μ_ξ} forcings:*

> *$\mathbb{P}(\bar{\mu})$ is the set of all functions p such that $\mathrm{dom}(p)$ is a subset of λ with cardinality $< \kappa$, and for every $\xi \in \mathrm{dom}(p)$, $p(\xi)$ is a condition in P_{μ_ξ};*

> *the order of $\mathbb{P}(\bar{\mu})$ is defined coordinate wise, i.e., for $p, q \in \mathbb{P}(\bar{\mu})$, $q \leq p$ if $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$ and for every $\xi \in \mathrm{dom}(p)$, $q(\xi) \leq p(\xi)$.*

The weakest condition in $\mathbb{P}(\bar{\mu})$ is the empty function, denoted by $\mathbf{1}$. For each $p \in \mathbb{P}(\bar{\mu})$ and $\xi \in \mathrm{dom}(p)$, we let the condition $p(\xi)$ be the pair $\langle T_\xi^p, \langle b_{\xi,\delta}^p \mid \delta \in \Delta_\xi^p \rangle \rangle$. From now on, Δ^p denotes the set $\{\langle \xi, \delta \rangle \mid \xi \in \mathrm{dom}(p) \text{ and } \delta \in \Delta_\xi^p\}$.

Fact 2.5

a) *The forcing $\mathbb{P}(\bar{\mu})$ is κ-closed and it has κ^+-c.c.*

b) *Suppose G is a $\mathbb{P}(\bar{\mu})$-generic set over V. In $V[G]$, for every $\xi < \lambda$, the κ-branches through the tree $T_\xi^G = \bigcup_{p \in G} T_\xi^p$ are $\{b_{\xi,\delta}^G \mid \delta < \mu_\xi\}$, where each $b_{\xi,\delta}^G$ is the function*

$$\bigcup \{b_{\xi,\delta}^p \mid p \in G, \xi \in \mathrm{dom}(p) \text{ and } \delta \in \Delta_\xi^p\}.$$

Proof. b) Since $\mathbf{1} \Vdash_{\mathbb{P}(\bar{\mu} \restriction (\xi+1))}$ "$\mathbb{P}(\bar{\mu} \restriction (\kappa \smallsetminus (\xi + 1)))$ is κ-closed", the claim follows from Lemma 2.3. $\blacksquare$

Definition 2.6 *For all $\mathbb{P}(\bar{\mu})$-names τ, define that*

$$\Delta^\tau = \bigcup\{\Delta^p \mid condition\ p\ appears\ in\ \tau\}.$$

Let Δ^τ_{1st} denote the set $\{\xi \mid \langle\xi,\delta\rangle \in \Delta^\tau\}$ and Δ^τ_ξ denote the set $\{\delta \mid \langle\xi,\delta\rangle \in \Delta^\tau\}$.

Definition 2.7 *Suppose $\bar{z} = \langle z_\xi \mid \xi \in Z\rangle$ is a sequence such that $Z \subseteq \lambda$ and for each $\xi \in Z$, z_ξ is a subset of μ_ξ of cardinality at least κ. In order to keep our notation coherent, let $\Delta^{\bar{z}}$ be a shorthand for the set $\bigcup_{\xi \in Z}\{\xi\} \times z_\xi$. We define*

$$\mathbb{P}(\bar{z}) = \{p \in \mathbb{P}(\bar{\mu}) \mid \Delta^p \subseteq \Delta^{\bar{z}}\}.$$

We say that $\mathbb{P}(\bar{z})$ is a subforcing of $\mathbb{P}(\bar{\mu})$ when $\bar{z}$ is a sequence as described above.

A forcing Q is *a complete subforcing of P* if every maximal antichain in Q is also a maximal antichain in P (a set X of conditions is an antichain in Y if all $p \neq q$ in X are incompatible, i.e., there is no $r \in Y$ with $r \leq p, q$). The following basic facts we need later on.

Fact 2.8

a) *Every subforcing $\mathbb{P}(\bar{z})$ is a complete subforcing of $\mathbb{P}(\bar{\mu})$.*

b) *For every $p \in \mathbb{P}(\bar{\mu})$, the restriction $\{q \in \mathbb{P}(\bar{\mu}) \mid q \leq p\}$ is a forcing notion which is equivalent to $\mathbb{P}(\bar{\mu})$.*

The following two definitions will be our main tools. Namely, every permutation π of the indices of the labels of the branches in the generic trees added by $\mathbb{P}(\bar{\mu})$ determines an automorphism $\hat{\pi}$ of $\mathbb{P}(\bar{\mu})$. This means that for every condition p in $\mathbb{P}(\bar{\mu})$ and $\mathbb{P}(\bar{\mu})$-name τ there are many "isomorphic" copies of p and τ inside $\mathbb{P}(\bar{\mu})$. Naturally, the copies $\hat{\pi}(p)$ and $\hat{\pi}(\tau)$ of p and τ, respectively, satisfies all the same formulas (see (1) below).

Definition 2.9 *We define $\mathrm{Mps}(\bar{\mu})$ to be the set of all functions π which can be defined as follows. The domain of π is $\Delta^{\bar{y}}$ for some sequence $\bar{y} = \langle y_\xi \mid \xi \in Y\rangle$ with $Y \subseteq \lambda$ and $y_\xi \subseteq \mu_\xi$ for each $\xi \in Y$. In addition, there*

*exists an injective function π_{1st} from Y into λ and injective functions π_ξ
from y_ξ into μ_ξ, for all $\xi \in Y$, such that for all $\langle \xi, \delta \rangle \in \text{dom}(\pi)$,*

$$\pi\langle \xi, \delta \rangle = \langle \pi_{1st}(\xi), \pi_\xi(\delta) \rangle.$$

When $\mathbb{P}(\bar{z})$ is a subforcing of $\mathbb{P}(\bar{\mu})$, let $\text{Mps}(\bar{z})$ be the collection $\{\pi \in \text{Mps}(\bar{\mu}) \mid \text{dom}(\pi) \subseteq \Delta^{\bar{z}}\}$.

Definition 2.10 *For every $p \in \mathbb{P}(\bar{\mu})$ and $\pi \in \text{Mps}(\bar{\mu})$ with $\Delta^p \subseteq \text{dom}(\pi)$,
we let $\pi(p)$ denote the condition q in $\mathbb{P}(\bar{\mu})$ for which*

$\text{dom}(q) = \pi_{1st}[\text{dom}(p)]$,

for every $\zeta \in \text{dom}(q)$, $T_\zeta^q = T_\xi^p$ and $\Delta_\zeta^q = \pi_\xi[\Delta_\xi^p]$, where $\xi = (\pi_{1st})^{-1}(\zeta)$;

for every $\langle \zeta, \varepsilon \rangle \in \Delta^q$, $b_{\zeta,\varepsilon}^q = b_{\xi,\delta}^p$, where $\langle \xi, \delta \rangle = \pi^{-1}\langle \zeta, \varepsilon \rangle$.

*When τ is a $\mathbb{P}(\bar{z})$-name and π a mapping in $\text{Mps}(\bar{\mu})$ with $\Delta^\tau \subseteq \text{dom}(\pi)$,
$\pi(\tau)$ denotes the $\mathbb{P}(\bar{z})$-name which is result of recursively replacing every
condition p in τ with $\pi(p)$, i.e.,*

$$\pi(\tau) = \{\langle \pi(\sigma), \pi(p) \rangle \mid \langle \sigma, p \rangle \in \tau\}.$$

*Analogously, for sequences $\bar{z} = \langle z_\xi \mid \xi \in Z \rangle$ with $\Delta^{\bar{z}} \subseteq \text{dom}(\pi)$, we let $\pi(\bar{z})$
denote the sequence $\langle z_\zeta' \mid \zeta \in Z' \rangle$, where $Z' = \pi_{1st}[Z]$ and for each $\zeta \in Z'$,
$z_\zeta' = \pi_\xi[z_\xi]$ with $\xi = (\pi_{1st})^{-1}(\zeta)$.*

Fact 2.11 *For every subforcing $\mathbb{P}(\bar{z})$ and $\pi \in \text{Mps}(\bar{z})$ with $\text{dom}(\pi) = \Delta^{\bar{z}}$,
the mapping $p \mapsto \pi(p)$ is an isomorphism between $\mathbb{P}(\bar{z})$ and $\mathbb{P}(\pi(\bar{z}))$.*

Suppose $\mathbb{P}(\bar{z})$ is a subforcing of $\mathbb{P}(\bar{\mu})$. The isomorphism determined by some
$\pi \in \text{Mps}(\bar{z})$ is denoted by $\hat{\pi}$. It follows that if $\text{dom}(\pi) = \Delta^{\bar{z}}$, $p \in \mathbb{P}(\bar{z})$,
$\psi(x_1, \ldots, x_n)$, $n < \omega$, is any formula, and $\tau_1, \ldots, \tau_n$ are $\mathbb{P}(\bar{z})$-names then

(1) $\qquad p \Vdash_{\mathbb{P}(\bar{z})} \psi(\tau_1, \ldots, \tau_n)$ iff $\hat{\pi}(p) \Vdash_{\mathbb{P}(\pi(\bar{z}))} \psi(\hat{\pi}(\tau_1), \ldots, \hat{\pi}(\tau_n))$.

Particularly, a mapping π in $\text{Mps}(\bar{z})$ determines an automorphism of $\mathbb{P}(\bar{z})$
when π_{1st} is a permutation of Z and each π_ξ is a bijection from z_ξ onto
$z_{\pi_{1st}(\xi)}$.

3 Basic facts on Σ_1^1-equivalence relations

Recall that we assumed κ to be an uncountable regular cardinal. We denote the set of all sets hereditarily of cardinality $< \kappa$ by $H(\kappa)$, i.e., $H(\kappa)$ contains all the sets whose transitive closure has cardinality $< \kappa$.

Definition 3.1 *We say that ϕ defines a Σ_1^1-equivalence relation $\sim_{\phi,R}$ on $^\kappa 2$ with a parameter $R \subseteq H(\kappa)$ when*

   a) *ϕ is a first order sentence in the vocabulary consisting of $\in$, one unary relation symbol S_0, and binary relation symbols S_1, S_2, and S_3;*

   b) *the following definition gives an equivalence relation on $^\kappa 2$: for all $f, g \in {}^\kappa 2$*

$$f \sim_{\phi,R} g \ iff \ \ for \ some \ h \in {}^\kappa 2, \langle H(\kappa), \in, R, f, g, h \rangle \models \phi,$$

   where R, f, g, and h are the interpretations of the symbols S_0, S_1, S_2, and S_3 respectively.

We abbreviate $\mathrm{card}\big(\{f/\sim_{\phi,R} \mid f \in {}^\kappa 2\}\big)$ by $\mathrm{No}(\sim_{\phi,R})$.

Lemma 3.2

   a) *For every nonzero cardinal $\mu \in \kappa \cup \{\kappa, 2^\kappa\}$, there exists a Σ_1^1-equivalence relation $\sim_{\phi,R}$ on $^\kappa 2$ with $\mathrm{No}(\sim_{\phi,R}) = \mu$.*

   b) *There exists a Σ_1^1-equivalence relation $\sim_{\phi,R}$ on $^\kappa 2$ with $\mathrm{No}(\sim_{\phi,R}) = \kappa^+$.*

   c) *If T is a tree with $\mathrm{card}(T) = \kappa$, then there exists a Σ_1^1-equivalence relation $\sim_{\phi,R}$ on $^\kappa 2$ with $\mathrm{No}(\sim_{\phi,R}) = \mathrm{card}(\mathrm{Br}_\kappa(T)) + 1$.*

Proof. Let ρ be a fixed definable bijection from κ onto $\kappa \times \kappa$. For a binary relation R, we denote the set $\{\rho(\xi) \mid$ for some $\xi < \kappa, \langle \xi, 1 \rangle \in R\}$ by $\rho(R)$.

a) In the cases $\mu \in \kappa \cup \{\kappa\}$, the parameter can code a list of μ-many nonequivalent functions. In the case $\mathrm{No}(\sim_{\phi,R}) = 2^\kappa$ all the functions in $^\kappa 2$ can be nonequivalent.

b) A sentence $\phi(R_1, R_2, R_3)$ saying

"(both $\rho(R_1)$ and $\rho(R_2)$ are well-orderings of κ, and $\rho(R_3)$ is an isomorphism between them) or (neither $\rho(R_1)$ nor $\rho(R_2)$ is a well-ordering of κ)"

defines a Σ_1^1-equivalence relation as wanted.

c) We may assume, without loss of generality, that the elements of T are ordinals below κ. Using $\langle T, < \rangle$ as a parameter, let a sentence $\phi(R_0, R_1, R_2)$ (see Definition 3.1) say that

"$(\rho(R_1) = \rho(R_2)$ is a κ-branch in R_0) or (neither $\rho(R_1)$ nor $\rho(R_2)$ is a κ-branch in R_0)".

Then ϕ defines a Σ_1^1-equivalence relation as wanted. ■3.2

Conclusion 3.3 *Let G be a $\mathbb{P}(\bar{\mu})$-generic set over V. Then in $V[G]$, for every nonzero cardinal χ in $\kappa \cup \{\kappa, \kappa^+, 2^\kappa\} \cup \{\mu_\xi \mid \xi < \lambda\}$, there exists a Σ_1^1-equivalence relation $\sim_{\phi,R}$ with $\mathrm{No}(\sim_{\phi,R}) = \chi$.*

Proof. The claim follows from Fact 2.5 together with Lemma 3.2. ■

In the next section we shall need the following properties of Σ_1^1-equivalence relations.

Lemma 3.4 *Suppose $\gamma \leq \kappa$ and χ_i, $i < \gamma$, are nonzero cardinals such that ϕ_i defines a Σ_1^1-equivalence relation on ${}^\kappa 2$ with the parameter R_i and it has exactly χ_i-many equivalence classes.*

  a) *There exists a Σ_1^1-equivalence relation $\sim_{\phi,R}$ on ${}^\kappa 2$ with $\mathrm{No}(\sim_{\phi,R}) = \bigcup_{i<\gamma} \chi_i$.*

  b) *There exists a Σ_1^1-equivalence relation $\sim_{\phi,R}$ on ${}^\kappa 2$ with $\mathrm{No}(\sim_{\phi,R}) = \mathrm{card}(\prod_{i<\gamma} \chi_i)$.*

Proof. Both of the claims are simple corollaries of the fact that there are a parameter $R \subseteq H(\kappa)$ and a formula $\psi(x)$ such that for all $f, g, h \in {}^\kappa 2$

$$\langle H(\kappa), \in, R, f, g, h \rangle \models \psi(i)$$

if, and only if,

$$\langle H(\kappa), \in, R[i], f[i], g[i], h[i] \rangle \models \phi_i,$$

where $R[i]$, $f[i]$, $g[i]$, and $h[i]$ are the ith parts of R, f, g, and h respectively, in some definable coding. Furthermore $R[i] = R_i$ holds for every $i < \gamma$. ■

10

Lemma 3.5 *Suppose that $\mathbb{P}(\bar{z})$ is a subforcing of $\mathbb{P}(\bar{\mu})$, ϕ is a sentence as in Definition 3.1(a), and $\tilde{R}$, σ_1, σ_2 are $\mathbb{P}(\bar{z})$-names of cardinality κ for subsets of $H(\kappa)$.*

a) *If $p \in \mathbb{P}(\bar{z})$ and ψ_1 denotes the sentence*

$$\text{``there is } h \in {}^\kappa 2 \text{ with } \langle H(\kappa), \in, \tilde{R}, \sigma_1, \sigma_2, h \rangle \models \phi\text{''},$$

   then $p \Vdash_{\mathbb{P}(\bar{z})} \psi_1$ implies that $p \Vdash_{\mathbb{P}(\bar{\mu})} \psi_1$ holds, too.

b) *(An auxiliary fact only applied in (c) of this lemma.) Suppose τ is a $\mathbb{P}(\bar{\mu})$-name of cardinality κ for a subset of $H(\kappa)$ and q is a condition in $\mathbb{P}(\bar{\mu})$ forcing that*

$$\langle H(\kappa), \in, \tilde{R}, \sigma_1, \sigma_2, \tau \rangle \models \phi.$$

   For any injection η from $\Delta^\tau_{1st} \smallsetminus Z$ into $\lambda \smallsetminus Z$ there is $\rho \in \mathrm{Mps}(\bar{\mu})$ satisfying that $\Delta^{\bar{z}} \cup \Delta^q \cup \Delta^\tau \subseteq \mathrm{dom}(\rho)$, $\rho{\restriction}\Delta^{\bar{z}}$ is identity, $\eta \subseteq \rho_{1st}$, and

$$\rho(q) \Vdash_{\mathbb{P}(\bar{\mu})} \langle H(\kappa), \in, \tilde{R}, \sigma_1, \sigma_2, \rho(\tau) \rangle \models \phi.$$

   Remark. *The conclusion holds even thought ρ does not determine an automorphism of $\mathbb{P}(\bar{\mu})$.*

c) *Suppose the length of $\bar{z}$ is at least κ^+, the cardinality of each z_ξ is at least κ^+, $p \in \mathbb{P}(\bar{z})$, and ψ_2 denotes the sentence*

$$\text{``for all } h \in {}^\kappa 2, \langle H(\kappa), \in, \tilde{R}, \sigma_1, \sigma_2, h \rangle \models \phi\text{''}.$$

   Then $p \Vdash_{\mathbb{P}(\bar{z})} \psi_2$ implies $p \Vdash_{\mathbb{P}(\bar{\mu})} \psi_2$.

d) *(An auxiliary fact only applied in Lemma 4.2.) Suppose τ is a $\mathbb{P}(\bar{\mu})$-name of cardinality κ for a subset of $H(\kappa)$ and q is a condition in $\mathbb{P}(\bar{\mu})$ forcing that*

$$\text{``for all } h \in {}^\kappa 2, \langle H(\kappa), \in, \tilde{R}, \sigma_1, \tau, h \rangle \models \phi\text{''}.$$

   Then for any injection ρ from $\Delta^\tau_{1st} \smallsetminus Z$ into $\lambda \smallsetminus Z$ there is $\pi \in \mathrm{Mps}(\bar{\mu})$ satisfying that $\Delta^{\bar{z}} \cup \Delta^q \cup \Delta^\tau \subseteq \mathrm{dom}(\rho)$, $\pi{\restriction}\Delta^{\bar{z}}$ is identity, $\rho \subseteq \pi_{1st}$, and

$$\pi(q) \Vdash_{\mathbb{P}(\bar{\mu})} \text{``for all } h \in {}^\kappa 2, \langle H(\kappa), \in, \tilde{R}, \sigma_1, \pi(\tau), h \rangle \models \phi\text{''}.$$

Proof. a) The claim follows from the facts that $\mathbb{P}(\bar{\mu})$ does not add new elements into $H(\kappa)$, and the truth in $H(\kappa)$ is absolute.

b) Define the subforcing $\mathbb{P}(\bar{y})$ to be "the smallest one" containing $\bar{z}$, q, and τ, i.e., define $\bar{y}$ to be the sequence $\langle y_\xi \mid \xi \in Y \rangle$ satisfying $\Delta^{\bar{y}} = \Delta^{\bar{z}} \cup \Delta^q \cup \Delta^\tau$. Since the truth in $H(\kappa)$ is absolute,

$$q \Vdash_{\mathbb{P}(\bar{y})} \langle H(\kappa), \in, \tilde{R}, \sigma_1, \sigma_2, \tau \rangle \models \phi.$$

Now each y_ξ, $\xi \notin Z$, is so small that there is $\rho \in \mathrm{Mps}(\bar{\mu})$ satisfying the demands: $\mathrm{dom}(\rho) = \Delta^{\bar{y}}$, $\rho_{1st} {\restriction} Z$ is identity, $\rho_{1st} {\restriction} (\Delta^\tau_{1st} \smallsetminus Z)$ is η, and for every $\xi \in Y$, ρ_ξ is identity if $\xi \in Z$, and otherwise, ρ_ξ is some injection from y_ξ into $\mu_{\eta(\xi)}$. Since ρ determines an isomorphism between $\mathbb{P}(\bar{y})$ and $\mathbb{P}(\rho(\bar{y}))$, we have that

$$\rho(q) \Vdash_{\mathbb{P}(\rho(\bar{y}))} \langle H(\kappa), \in, \tilde{R}, \sigma_1, \sigma_2, \rho(\tau) \rangle \models \phi.$$

Again, by the absoluteness of the truth in $H(\kappa)$, we can conclude that the condition $\rho(q)$ forces the same sentence in the larger forcing $\mathbb{P}(\bar{\mu})$.

c) Assume, contrary to the claim, that p is a condition in $\mathbb{P}(\bar{z})$ forcing ψ_2, $q \leq p$ is a condition in $\mathbb{P}(\bar{\mu})$, and τ is a $\mathbb{P}(\bar{\mu})$-name for a function from κ into 2 so that

(A) $$q \Vdash_{\mathbb{P}(\bar{\mu})} \left(\langle H(\kappa), \in, \tilde{R}, \sigma_1, \sigma_2, \tau \rangle \not\models \phi \right).$$

We shall define a mapping $\pi \in \mathrm{Mps}(\bar{\mu})$ such that it determines an automorphism $\hat{\pi}$ of $\mathbb{P}(\bar{\mu})$ with the following properties: $\hat{\pi}(\tilde{R}) = \tilde{R}$, $\hat{\pi}(\sigma_1) = \sigma_1$, $\hat{\pi}(\sigma_2) = \sigma_2$, $\hat{\pi}(\tau)$ is a $\mathbb{P}(\bar{z})$-name, $\hat{\pi}(q) \in \mathbb{P}(\bar{z})$, and $\hat{\pi}(q)$ is compatible with q. It follows that $\hat{\pi}(q) \Vdash_{\mathbb{P}(\bar{\mu})} \left(\langle H(\kappa), \in, \tilde{R}, \sigma_1, \sigma_2, \hat{\pi}(\tau) \rangle \not\models \phi \right)$. Then, by the absoluteness of truth in $H(\kappa)$, we have that

$$\hat{\pi}(q) \Vdash_{\mathbb{P}(\bar{z})} \left(\langle H(\kappa), \in, \tilde{R}, \sigma_1, \sigma_2, \hat{\pi}(\tau) \rangle \not\models \phi \right).$$

Since there exist $r \in \mathbb{P}(\bar{\mu})$ with $r \leq p$ and $r \leq \hat{\pi}(q)$, we have a contradiction.

We need the demands that $\mathrm{card}(Z) > \kappa$ and $\mathrm{card}(z_\xi) > \kappa$, $\xi \in Z$, to ensure that $Z \smallsetminus (\Delta^{\sigma_1} \cup \Delta^{\sigma_2})$ and each $z_\xi \smallsetminus (\Delta^{\sigma_1}_\xi \cup \Delta^{\sigma_2}_\xi)$ has cardinality $\geq \kappa$ (otherwise it is difficult to choose π satisfying both $\pi(\tau) \subseteq \Delta^{\bar{z}}$ and $\pi {\restriction} (\Delta^{\sigma_1} \cup \Delta^{\sigma_2})$ is identity).

Remark. A mapping $\pi \in \mathrm{Mps}(\bar{\mu})$ determines an automorphism if π_{1st} is a permutation of λ and each π_ξ is a bijection from μ_ξ onto $\mu_{\pi_{1st}(\xi)}$. Thus

we need that the chosen π satisfies $\mu_{\pi_{1st}(\xi)} = \mu_\xi$ for every $\xi \in \Delta^\tau_{1st}$. Now Z might be too small to contain all the possible cardinals in $\bar\mu$. However, because of (b) and the assumption that all the cardinals in $\bar\mu$ are repeated λ-many times, there are q' and τ' satisfying (A). Moreover, for every $\xi \in \Delta^\tau_{1st} \smallsetminus Z$, there is $\zeta \in Z \smallsetminus (\Delta^q_{1st} \cup \Delta^{\tilde R}_{1st} \cup \Delta^{\sigma_1}_{1st} \cup \Delta^{\sigma_2}_{1st})$ with $\mu_\zeta = \mu_\xi$, and $q' \leq p$ holds, too (since $q' = \rho(q)$ and the branches in p are kept fixed, i.e., $\rho{\restriction}\Delta^{\bar z}$ is identity).

Define, for every $\langle \xi, \delta \rangle \in \Delta^q \cup \Delta^\tau$, a pair $\langle \zeta_\xi, \varepsilon_{\xi,\delta} \rangle$ as follows. Set $\zeta_\xi = \xi$ if $\xi \in \Delta^{\tilde R}_{1st} \cup \Delta^{\sigma_1}_{1st} \cup \Delta^{\sigma_2}_{1st}$, and choose $\zeta_\xi \in Z \smallsetminus (\Delta^q_{1st} \cup \Delta^{\tilde R}_{1st} \cup \Delta^{\sigma_1}_{1st} \cup \Delta^{\sigma_2}_{1st})$ with $\mu_{\zeta_\xi} = \mu_\xi$ otherwise. Analogously, let $\varepsilon_{\xi,\delta}$ be δ if δ is in $\Delta^{\tilde R}_\xi \cup \Delta^{\sigma_1}_\xi \cup \Delta^{\sigma_2}_\xi$, and pick some $\varepsilon_{\xi,\delta}$ from $z_\xi \smallsetminus (\Delta^q_\xi \cup \Delta^{\tilde R}_\xi \cup \Delta^{\sigma_1}_\xi \cup \Delta^{\sigma_2}_\xi)$ otherwise. Let π be any mapping from $\mathrm{Mps}(\bar\mu)$ which satisfies that

$\pi{\restriction}(\Delta^{\tilde R} \cup \Delta^{\sigma_1} \cup \Delta^{\sigma_2})$ is identity;

for every $\langle \xi, \delta \rangle \in \Delta^q \cup \Delta^\tau$, $\pi\langle \xi, \delta \rangle = \langle \zeta_\xi, \varepsilon_{\xi,\delta} \rangle$.

π_{1st} is a permutation of λ;

for every $\xi < \lambda$, π_ξ is a permutation of μ_ξ;

Then π determines an automorphism as wanted.

d) The proof is similar to the proof of (b). The main difference is that one has to apply (a) and (c) instead of "the absoluteness of truth in $H(\kappa)$". $\blacksquare$3.5

Conclusion 3.6 *Suppose $\mathbb{P}(\bar z)$ is a subforcing of $\mathbb{P}(\bar\mu)$ and ψ is a sentence in the vocabulary $\{\in, S_0, S_1, S_2\}$ which is a Boolean combination of a sentence containing one second order existential quantifier and a sentence containing one second order universal quantifier. Then for every $\mathbb{P}(\bar\mu)$-generic set H over V, $G = H \cap \mathbb{P}(\bar z)$ is $\mathbb{P}(\bar z)$-generic set over V, $V[G] \subseteq V[H]$, and for all $f, g \in ({}^\kappa 2)^{V[G]}$,*

$$ (\langle H(\kappa), \in, R, f, g \rangle \models \psi)^{V[G]} \text{ iff } (\langle H(\kappa), \in, R, f, g \rangle \models \psi)^{V[H]}, $$

where R, f, and g are interpretations of the symbols S_0, S_1, and S_2 respectively.

4 Possible numbers of equivalence classes

Definition 4.1 *Suppose $\bar{\mu} = \langle \mu_\xi \mid \xi < \lambda \rangle$ is a sequence of cardinals. Define $\Omega_{\bar{\mu}}$ to be the smallest set of cardinals satisfying that*

> *every nonzero cardinal $\leq \kappa^+$ is in $\Omega_{\bar{\mu}}$;*

> *$\{\mu_\xi \mid \xi < \lambda\} \subseteq \Omega_{\bar{\mu}}$;*

> *if $\gamma \leq \kappa$ and χ_i, $i < \gamma$, are cardinals in $\Omega_{\bar{\mu}}$, then both $\bigcup_{i<\gamma} \chi_i$ and $\mathrm{card}(\prod_{i<\gamma} \chi_i)$ are in $\Omega_{\bar{\mu}}$.*

We shall now prove that, when the singular cardinal hypothesis holds, the closure under unions and products as above are, consistent wise, the only restrictions on the possible numbers of equivalence classes of Σ_1^1-equivalence relations.

Theorem 1 *Suppose that*

> *κ is an uncountable cardinal with $\kappa^{<\kappa} = \kappa$ and $2^\kappa = \kappa^+$;*

> *$\lambda > \kappa^+$ is a cardinal with $\lambda^\kappa = \lambda$;*

> *$\bar{\mu} = \langle \mu_\xi \mid \xi < \lambda \rangle$ and $\mathbb{P}(\bar{\mu})$ are as in Definition 2.4;*

> *$\Omega_{\bar{\mu}}$ is as in Definition 4.1;*

> *for every $\chi \in \Omega_{\bar{\mu}}$ with $\chi > \kappa^+$ and $\gamma < \kappa$, the inequality $\chi^\gamma \leq \chi^+$ holds.*

Then for every $\mathbb{P}(\bar{\mu})$-generic set G, the extension $V[G]$ satisfies that all cardinals and cofinalities are preserved, there are no new sets of cardinality $< \kappa$, $2^\kappa = \lambda$ and for all cardinals χ, the following two conditions are equivalent:

a) *$\chi \in \Omega_{\bar{\mu}}$;*

b) *a sentence ϕ defines a Σ_1^1-equivalence relation $\sim_{\phi,R}$ on $^\kappa 2$ with a parameter $R \subseteq H(\kappa)$ and there are exactly χ different equivalence classes of $\sim_{\phi,R}$.*

The rest of this section is devoted to the proof of this theorem. Because of Conclusion 3.3 and Lemma 3.4 it remains to show that

$$\mathbf{1} \Vdash_{\mathbb{P}(\bar{\mu})} \text{``No}(\sim_{\phi,\tilde{S}}) \in \Omega_{\bar{\mu}} \text{ for all } \Sigma^1_1\text{-equivalence relations } \sim_{\phi,\tilde{S}}\text{''}.$$

Suppose p is a condition in $\mathbb{P}(\bar{\mu})$ and θ a cardinal such that

$$p \Vdash_{\mathbb{P}(\bar{\mu})} \text{``there exists } X \subseteq H(\kappa) \text{ with No}(\sim_{\phi,X}) = \theta\text{''}.$$

By Fact 2.8(b) we have that the condition $\mathbf{1}$ forces the same formula. Hence by the maximal principle we may fix a name $\tilde{R}$ so that $\mathbf{1} \Vdash_{\mathbb{P}(\bar{\mu})} \tilde{R} \subseteq H(\kappa)$ and

(2) $$\mathbf{1} \Vdash_{\mathbb{P}(\bar{\mu})} \text{No}(\sim_{\phi,\tilde{R}}) = \theta.$$

Since $\mathbb{P}(\bar{\mu})$ has κ^+-c.c. and $\text{card}(H(\kappa)) = \kappa^{<\kappa} = \kappa$, we may assume that the name $\tilde{R}$ has cardinality κ.

4.1 Choice of a small subforcing

Next we want to prove that there is a subforcing $\mathbb{P}(\bar{z})$ of $\mathbb{P}(\bar{\mu})$ such that the cardinality of $\mathbb{P}(\bar{z})$ is θ, there are only κ^+-many coordinates in $\mathbb{P}(\bar{z})$, and already $\mathbb{P}(\bar{z})$ produces θ-many different equivalent classes of $\sim_{\phi,\tilde{R}}$.

Lemma 4.2 *Suppose $\mathbb{P}(\bar{z})$ is a subforcing of $\mathbb{P}(\bar{\mu})$ such that*

$\bar{z} = \langle z_\xi \mid \xi \in Z \rangle$;

Z is a subset of λ satisfying $\text{card}(Z) = \kappa^+$ and $\Delta^{\tilde{R}}_{1st} \subseteq Z$;

for each $\xi \in \Delta^{\tilde{R}}_{1st}$, $z_\xi = \mu_\xi$ if $\mu_\xi \leq \theta$, and otherwise, $z_\xi \in \{y \in [\mu_\xi]^{\kappa^+} \mid \Delta^{\tilde{R}}_\xi \subseteq y\}$.

$Z \smallsetminus \Delta^{\tilde{R}}_{1st}$ is of cardinality κ^+;

for every $\xi \in Z \smallsetminus \Delta^{\tilde{R}}_{1st}$, $\mu_\xi > \theta$ and z_ξ is some set in $[\mu_\xi]^{\kappa^+}$.

Then our assumption (2) on this page implies $\mathbf{1} \Vdash_{\mathbb{P}(\bar{z})} \text{No}(\sim_{\phi,\tilde{R}}) = \theta$.

Proof. Let $\tilde{\mathcal{F}}_{\bar{z}}$ be a $\mathbb{P}(\bar{z})$-name for the set of all functions from κ into 2, i.e., it satisfies that $\mathbf{1} \Vdash_{\mathbb{P}(\bar{z})} \tilde{\mathcal{F}}_{\bar{z}} = {}^\kappa 2$. We prove that

(A) $\qquad \mathbf{1} \Vdash_{\mathbb{P}(\bar{\mu})}$ "for all $f \in {}^\kappa 2$ there is $g \in \tilde{\mathcal{F}}_{\bar{z}}$ with $f \sim_{\phi,\tilde{R}} g$".

This suffices since then

$$\mathbf{1} \Vdash_{\mathbb{P}(\bar{\mu})} \theta \leq \mathrm{card}(\tilde{\mathcal{F}}_{\bar{z}}/\!\!\sim_{\phi,\tilde{R}}) \leq \mathrm{card}({}^{\kappa}2/\!\!\sim_{\phi,\tilde{R}}) = \mathrm{No}(\sim_{\phi,\tilde{R}}) = \theta,$$

and, by (a), (c) of Lemma 3.5, we can conclude

$$\mathbf{1} \Vdash_{\mathbb{P}(\bar{z})} \mathrm{No}(\sim_{\phi,\tilde{R}}) = \mathrm{card}(\tilde{\mathcal{F}}_{\bar{z}}/\!\!\sim_{\phi,\tilde{R}}) = \theta.$$

Now assume, contrary to (A), that (2) on the page before holds, there is a condition p in $\mathbb{P}(\bar{\mu})$, and a $\mathbb{P}(\bar{\mu})$-name σ for a function from κ into 2 such that

(B) $$p \Vdash_{\mathbb{P}(\bar{\mu})} \text{``for all } g \in \tilde{\mathcal{F}}_{\bar{z}}, \sigma \not\sim_{\phi,\tilde{R}} g \text{''}.$$

We may assume that the name σ is of cardinality κ. Furthermore, by Lemma 3.5(d) and since each cardinal in $\bar{\mu}$ is listed λ-many times, we may choose p and the name σ so that the coordinates appearing in σ adds a tree with the same number of κ-branches as some coordinate in Z does, i.e., for every $\xi \in \Delta^{\sigma}_{1st}$, there is $\zeta \in Z$ with $\mu_{\zeta} = \mu_{\xi}$. This property will be essential in the choice of automorphisms (in the same way as the analogous demand Lemma 3.5(b) was needed in the proof of Lemma 3.5(c), see the remark in the middle of that proof).

Our strategy will be the following.

a) We define a name σ' so that $\mathbf{1} \Vdash_{\mathbb{P}(\bar{\mu})} \sigma' \in \tilde{\mathcal{F}}_{\bar{z}}$. Hence, by applying (B), we get

$$p \Vdash_{\mathbb{P}(\bar{\mu})} \sigma \not\sim_{\phi,\tilde{R}} \sigma'.$$

b) We define $\mathbb{P}(\bar{\mu})$-names $\langle \tau^{\gamma} \mid \gamma < \theta^{+} \rangle$ for functions from κ into 2, and conditions $\langle q^{\gamma} \mid \gamma < \theta^{+} \rangle$ in $\mathbb{P}(\bar{\mu})$.

c) For every $\gamma < \gamma' < \theta^{+}$ we define a mapping $\rho^{\gamma,\gamma'}$ in $\mathrm{Mps}(\bar{\mu})$ such that $\rho^{\gamma,\gamma'}$ determines an automorphism $\hat{\rho}^{\gamma,\gamma'}$ of $\mathbb{P}(\bar{\mu})$ with the following properties: $\hat{\rho}^{\gamma,\gamma'}(\tilde{R}) = \tilde{R}$, $\hat{\rho}^{\gamma,\gamma'}(p) = q^{\gamma}$, $\hat{\rho}^{\gamma,\gamma'}(\sigma) = \tau^{\gamma}$, and $\hat{\rho}^{\gamma,\gamma'}(\sigma') = \tau^{\gamma'}$. Hence it follows from (a) that

$$q^{\gamma} \Vdash_{\mathbb{P}(\bar{\mu})} \tau^{\gamma} \not\sim_{\phi,\tilde{R}} \tau^{\gamma'}.$$

d) Finally, we fix a $\mathbb{P}(\bar{\mu})$-generic set G over V and, by applying "a standard density argument", we show that for some $B \in [\theta^{+}]^{\theta^{+}}$, all the conditions q^{γ}, $\gamma \in B$, are in the generic set G. It follows from (c) that in $V[G]$, $\mathrm{No}(\sim_{\phi,R}) \geq \theta^{+}$ contrary to (2) on the preceding page.

16

As can be guessed from the demands on the sequence $\bar{z}$, there are three different kind of indices which we have to deal with:

$$\Theta_{\leq} = \{\xi \in \Delta_{1st}^{\tilde{R}} \mid \mu_{\xi} \leq \theta\},$$

$$\Theta_{>} = \{\xi \in \Delta_{1st}^{\tilde{R}} \mid \mu_{\xi} > \theta\}, \text{ and}$$

$$\Theta' = \lambda \smallsetminus \Delta_{1st}^{\tilde{R}}.$$

Remark. Of course we would like to have that $q^{\gamma} = \rho^{\gamma,\gamma'}(p) = p$ for every $\gamma < \gamma' < \theta^+$. Unfortunately, that is not possible since it might be the case that for some $\xi \in \Theta_{>}$, $\Delta_{\xi}^{\sigma} \cap \Delta_{\xi}^{p} \not\subseteq z_{\xi}$ (and we really need later the restriction $\operatorname{card}(z_{\xi}) < \theta$).

a) We define the name σ' to be $\pi(\sigma)$ for a mapping π in $\operatorname{Mps}(\bar{\mu})$ which satisfies the following conditions:

$\operatorname{dom}(\pi) = \Delta^{\sigma}$;

$\operatorname{ran}(\pi) \subseteq \Delta^{\bar{z}}$;

for every $\xi \in \operatorname{dom}(\pi_{1st})$, $\mu_{(\pi_{1st}(\xi))} = \mu_{\xi}$;

$\pi \restriction \Delta^{\tilde{R}}$ is identity (implying $\pi_{1st} \restriction (\Theta_{\leq} \cup \Theta_{>})$ is identity);

for every $\xi \in \operatorname{dom}(\pi_{1st}) \cap \Theta_{\leq}$, π_{ξ} is identity;

for every $\xi \in \operatorname{dom}(\pi_{1st}) \cap \Theta_{>}$ and $\delta \in \operatorname{dom}(\pi_{\xi}) \smallsetminus \Delta_{\xi}^{\tilde{R}}$, $\pi_{\xi}(\delta) \notin \Delta_{\xi}^{p} \cup \Delta_{\xi}^{\tilde{R}} \cup \Delta_{\xi}^{\sigma}$;

for every $\xi \in \operatorname{dom}(\pi_{1st}) \cap \Theta'$, $\pi_{1st}(\xi) \notin \operatorname{dom}(p) \cup \Delta_{1st}^{\tilde{R}} \cup \Delta_{1st}^{\sigma}$ and π_{ξ} is some injective function having range z_{ξ}.

It is possible to fulfill these conditions by the choice of σ, because of the cardinality demands on $\bar{z}$, and since $\Delta^p \cup \Delta^{\tilde{R}} \cup \Delta^{\sigma}$ has cardinality κ. Since $\mathbf{1} \Vdash_{\mathbb{P}(\bar{\mu})} \sigma \in {}^{\kappa}2$ and π can be extended so that the extension determines an automorphism of $\mathbb{P}(\bar{\mu})$, we have that $\mathbf{1} \Vdash_{\mathbb{P}(\bar{\mu})} \sigma' \in {}^{\kappa}2$. However, σ' is a $\mathbb{P}(\bar{z})$-name, so $\mathbf{1} \Vdash_{\mathbb{P}(\bar{\mu})} \sigma' \in \tilde{\mathcal{F}}_{\bar{z}}$ holds, too.

b) For every $\gamma < \theta^+$, we define a mapping $\pi^{\gamma} \in \operatorname{Mps}(\bar{\mu})$ so that the desired name τ^{γ} is $\pi^{\gamma}(\sigma)$ and the condition q^{γ} is $\pi^{\gamma}(p)$. Since we do NOT demand that $\operatorname{ran}(\pi^{\gamma}) \subseteq \Delta^{\bar{z}}$, when $\gamma < \theta^+$, it is possible to choose π^{γ} so that all the following demands are fulfilled:

17

$$\mathrm{dom}(\pi^\gamma) = \Delta^\sigma \cup \Delta^p$$

$\pi^\gamma {\restriction} \Delta^{\tilde{R}}$ is identity;

for every $\xi \in \mathrm{dom}(\pi^\gamma_{1st})$, $\mu_{(\pi^\gamma_{1st}(\xi))} = \mu_\xi$;

for every $\xi \in \mathrm{dom}(\pi^\gamma_{1st}) \cap (\Theta_\le \cup \Theta')$, π^γ_ξ is identity;

for all $\xi \in \mathrm{dom}(\pi^\gamma_{1st}) \cap \Theta_>$, the sets $(\Delta^{\tilde{R}}_\xi \cup \Delta^\sigma_\xi \cup \Delta^{\sigma'}_\xi \cup \Delta^p_\xi)$ and $\mathrm{ran}(\pi^\gamma_\xi) \smallsetminus \Delta^{\tilde{R}}_\xi$, for all $\gamma < \theta^+$, are pairwise disjoint;

the sets $(\Delta^{\tilde{R}}_{1st} \cup \Delta^\sigma_{1st} \cup \Delta^{\sigma'}_{1st} \cup \mathrm{dom}(p))$ and $\mathrm{ran}(\pi^\gamma_{1st}) \smallsetminus \Delta^{\tilde{R}}_{1st}$, for all $\gamma < \theta^+$, are pairwise disjoint.

c) Fix indices $\gamma < \gamma' < \theta^+$. Consider the set of pairs $\langle x, y \rangle$ satisfying that

$x \in \mathrm{dom}(\pi^\gamma)$ and $\pi^\gamma(x) = y$, or

there is $z \in \mathrm{dom}(\pi) = \Delta^\sigma$ such that $\pi(z) = x$ and $\pi^{\gamma'}(z) = y$.

Because of the conditions given above, we have that

for all $\xi \in \mathrm{dom}(\pi^\gamma_{1st}) = \mathrm{dom}(\pi^{\gamma'}_{1st})$, $\pi^\gamma_{1st}(\xi) = \pi^{\gamma'}_{1st}(\xi)$ iff $\pi_{1st}(\xi) = \xi$;

for all $\langle \xi, \delta \rangle \in \mathrm{dom}(\pi^\gamma) = \mathrm{dom}(\pi^{\gamma'})$, $\pi^\gamma_\xi(\delta) = \pi^{\gamma'}_\xi(\delta)$ iff $\pi_\xi(\delta) = \delta$;

for all $\xi \ne \zeta \in \mathrm{dom}(\pi^\gamma_{1st})$, $\pi^\gamma_\xi(\delta) \ne \pi^{\gamma'}_\zeta(\varepsilon)$;

for all $\langle \xi, \delta \rangle \ne \langle \xi, \varepsilon \rangle \in \mathrm{dom}(\pi^\gamma)$, $\pi^\gamma_\xi(\delta) \ne \pi^{\gamma'}_\xi(\varepsilon)$.

Hence the set of pairs we considered is the following well-defined injective function from $\mathrm{Mps}(\bar{\mu})$:

$$\eta = \pi^\gamma \cup \left((\pi^{\gamma'} {\restriction} \mathrm{dom}(\pi)) \circ (\pi)^{-1} \right).$$

We let the mapping $\rho^{\gamma,\gamma'}$ be any extension of η satisfying that $\rho^{\gamma,\gamma'} \in \mathrm{Mps}(\bar{\mu})$, $\mathrm{dom}(\rho^{\gamma,\gamma'}_{1st}) = \lambda$, and for each $\xi < \lambda$, $\mathrm{dom}(\rho^{\gamma,\gamma'}_\xi) = \mu_\xi$. It follows that

$$\rho^{\gamma,\gamma'}(\tilde{R}) = \pi(\tilde{R}) = \pi^\gamma(\tilde{R}) = \pi^{\gamma'}(\tilde{R}) = \tilde{R};$$

$$\rho^{\gamma,\gamma'}(p) = \pi^\gamma(p) = q^\gamma \text{ (note, that } \mathrm{ran}(\pi) \cap (\Delta^p \smallsetminus \Delta^{\tilde{R}}) = \emptyset);$$

$$\rho^{\gamma,\gamma'}(\sigma) = \pi^\gamma(\sigma) = \tau^\gamma \text{ (note, that } \operatorname{ran}(\pi) \cap (\Delta^\sigma \smallsetminus \Delta^{\tilde{R}}) = \emptyset);$$

$$\rho^{\gamma,\gamma'}(\sigma') = \pi^{\gamma'}\left(\pi^{-1}(\sigma')\right) = \pi^{\gamma'}(\sigma) = \tau^{\gamma'} \text{ (remember, that } \pi(\sigma) = \sigma').$$

d) Our demands on the mappings π^γ, $\gamma < \theta^+$, ensure that for each $\langle \xi, \delta \rangle \in \Delta^p$, if $\langle \xi, \delta \rangle \in \Delta^{(q^\gamma)}$ then $b_{\xi,\delta}^{(q^\gamma)} = b_{\xi,\delta}^p$. Therefore, p and q^γ are compatible conditions. Moreover, for every $\beta < \theta^+$, the set

$$D_\beta = \{ r \in \mathbb{P}(\bar{\mu}) \mid \text{for some } \gamma > \beta, r \leq q^\gamma \}$$

is a dense set below the condition p (which means that for every $s \leq p$ there is $r \leq s$ with $r \in D_\beta$). Because of $p \in G$, $D_\beta \cap G$ is nonempty for every $\beta < \theta^+$. Consequently, the set $B = \{ \gamma < \theta^+ \mid q^\gamma \in G \}$ must be cofinal in θ^+. So B has cardinality θ^+. ∎4.2

4.2 Isomorphism classes of names

First of all we fix $\bar{z}$ so that the subforcing $\mathbb{P}(\bar{z})$ of $\mathbb{P}(\bar{\mu})$ satisfies the assumptions of Lemma 4.2. Secondly we fix $\mathbb{P}(\bar{z})$-names $\langle \sigma_\alpha \mid \alpha < \theta \rangle$ for functions from κ into 2 so that for all $\alpha \neq \beta < \theta$,

(3) $$\mathbf{1} \Vdash_{\mathbb{P}(\bar{z})} \sigma_\alpha \not\sim_{\phi,\tilde{R}} \sigma_\beta.$$

Since $\mathbb{P}(\bar{z})$ has κ^+-c.c., we may assume that each of the names σ_α has cardinality κ.

Definition 4.3 *For every $\alpha < \theta$ we fix an enumeration $\langle \langle \xi_i^\alpha, \delta_i^\alpha \rangle \mid i < \kappa \rangle$ of Δ^{σ_α} without repetition. Names σ_α and σ_β are said to be isomorphic, written $\sigma_\alpha \cong \sigma_\beta$, if the following conditions are met:*

for every $i < \kappa$, $\xi_i^\alpha = \xi_i^\beta$;

for every $i < \kappa$ and $\zeta = \xi_i^\alpha = \xi_i^\beta$, if $\mu_\zeta > \theta$ then also $\delta_i^\alpha = \delta_i^\beta$;

for all $\langle \zeta, \varepsilon \rangle \in \Delta^{\tilde{R}}$ and $i < \kappa$, $\langle \xi_i^\alpha, \delta_i^\alpha \rangle = \langle \zeta, \varepsilon \rangle$ iff $\langle \xi_i^\beta, \delta_i^\beta \rangle = \langle \zeta, \varepsilon \rangle$.

$\pi(\sigma_\alpha) = \sigma_\beta$ when $\pi \in \operatorname{Mps}(\bar{z})$ is the mapping with $\operatorname{dom}(\pi) = \Delta^{\sigma_\alpha}$ and $\pi \langle \xi_i^\alpha, \delta_i^\alpha \rangle = \langle \xi_i^\beta, \delta_i^\beta \rangle$ for each $i < \kappa$.

For every $\alpha < \theta$ we denote the set $\{\beta < \theta \mid \sigma_\beta \cong \sigma_\alpha\}$ by Λ^α. Now by the choice of $\mathbb{P}(\bar{z})$, and the assumptions $\kappa^{<\kappa} = \kappa$ and $2^\kappa = \kappa^+$, the number of nonisomorphic names in $\{\sigma_\alpha \mid \alpha < \theta\}$ is $\leq \kappa^+$, i.e., the cardinality of the family $\{\Lambda^\alpha \mid \alpha < \theta\}$ is at most κ^+.

Let Γ be a subset of θ such that $\operatorname{card}(\Gamma) \leq \kappa^+$ and $\{\sigma_\alpha \mid \alpha \in \Gamma\}$ is a set of representatives of the isomorphism classes. If $\theta \leq \kappa^+$ then $\theta \in \Omega_{\bar{\mu}}$ directly by the definition. From now on we assume that $\theta > \kappa^+$. Hence $\theta = \bigcup_{\alpha \in \Gamma} \Lambda^\alpha$ implies that

$$(4) \qquad \theta = \bigcup_{\alpha \in \Gamma} \operatorname{card}(\Lambda^\alpha).$$

Define "the set of all small cardinals" to be

$$S(\tilde{R}) = \{\mu_\xi \mid \xi \in \Delta_{1st}^{\tilde{R}} \text{ and } \mu_\xi \leq \theta\}.$$

Note that this set might be empty. Anyway, then we know that

$$(5) \qquad \theta \geq \max\{\kappa^{++}, \sup S(\tilde{R})\}.$$

So to prove that θ is a cardinal in $\Omega_{\bar{\mu}}$ we shall show that for every $\alpha \in \Gamma$, the cardinality of Λ^α is strictly smaller than the lower bound given in (5) above, or otherwise, we can find a subset I^α of κ so that $\operatorname{card}(\Lambda^\alpha)$ has one of the following form: either $\operatorname{card}(I^\alpha) < \kappa$ and

$$(6) \qquad \operatorname{card}(\Lambda^\alpha) \in \Big\{ \bigcup_{i \in I^\alpha} \mu_{\xi_i^\alpha} \Big\} \cup \Big\{ \prod_{i \in I^\alpha} \mu_{\xi_i^\alpha} \Big\},$$

or else, $\operatorname{card}(I^\alpha) = \kappa$ and

$$(7) \qquad \operatorname{card}(\Lambda^\alpha) = \bigcup_{K \in [I^\alpha]^{<\kappa}} \operatorname{card}\Big(\prod_{i \in K} \mu_{\xi_i^\alpha} \Big).$$

This will suffice since we take care of that for every $\alpha \in \Gamma$ and for each $i \in I^\alpha$, $\mu_{\xi_i^\alpha} \in S(\tilde{R})$, i.e., only those small cardinals are used whose coordinate appears in the name $\tilde{R}$. Then there occurs at most κ-many different cardinals in the union (4), and hence, for some sequence $\langle X_k \mid k < \kappa \rangle$ of sets in $[S(\tilde{R})]^{<\kappa}$,

$$\theta = \bigcup_{k < \kappa} \operatorname{card}\Big(\prod_{\mu \in X_k} \mu \Big) \in \Omega_{\bar{\mu}}.$$

Remark. From our assumption that for every $\chi \in (\Omega_{\bar{\mu}} \setminus \kappa^{++})$ and $\gamma < \kappa$, the inequality $\chi^\gamma \leq \chi^+$ holds, it follows that θ is either $\sup S(\tilde{R})$ or $\operatorname{card}\Big(\prod_{\mu \in X} \mu \Big)$ for some subset X of $S(\tilde{R})$ with $\operatorname{card}(X) < \kappa$.

4.3 Case 1: The parameter depends on $< \kappa$-many coordinates

For the rest of the proof, let α^* be a fixed ordinal so that the number of names in $\{\sigma_\beta \mid \beta < \theta\}$, which are isomorphic to the representative σ_{α^*}, is greater or equal to the lower bound given in (5) on the preceding page, i.e., $\alpha^* \in \Gamma$ and $\mathrm{card}(\Lambda^{\alpha^*})$ is large enough. To simplify our notation, let $\bar{\xi}^* = \langle \xi_i^* \mid i < \kappa \rangle$ and $\bar{\delta}^* = \langle \delta_i^* \mid i < \kappa \rangle$ denote the sequences $\bar{\xi}^{\alpha^*}$ and $\bar{\delta}^{\alpha^*}$ respectively, and abbreviate Λ^{α^*} by Λ^*.

Define the set of "all critical indices of the isomorphism class of σ_{α^*}" to be

$$(8) \qquad J^* = \{i < \kappa \mid \mu_{\xi_i^*} \le \theta \text{ and } \langle \xi_i^*, \delta_i^* \rangle \notin \Delta^{\tilde{R}}\}.$$

Note that for every $\alpha \in \Lambda^*$, the equations $\bar{\xi}^\alpha = \bar{\xi}^*$ and $\bar{\delta}^\alpha {\restriction} (\kappa \smallsetminus J^*) = \bar{\delta}^* {\restriction} (\kappa \smallsetminus J^*)$ hold. Note also, that by the choice of $\mathbb{P}(\bar{z})$, $J^* \subseteq \{i < \kappa \mid z_{\xi_i^*} = \mu_{\xi_i^*}\} \subseteq \{i < \kappa \mid \xi_i^* \in \Delta_{1st}^{\tilde{R}}\}$. Thus $\{\mu_{\xi_i^*} \mid i \in J^*\} \subseteq S(\tilde{R})$ holds, too.

The set J^* must be nonempty, since otherwise there are $\alpha \ne \beta$ in Λ^* such that σ_α is the same name as σ_β, contrary to the choice that σ_α and σ_β are names for nonequivalent functions ((3) on page 19). For a similar reason $\mathrm{card}\big(\prod_{i \in J^*} \mu_{\xi_i^*}\big) \ge \mathrm{card}(\Lambda^*)$ holds.

Now suppose that already some subset K of J^* having cardinality $< \kappa$ satisfies the following inequality:

$$\mathrm{card}\big(\prod_{i \in K} \mu_{\xi_i^*}\big) \ge \mathrm{card}(\Lambda^*).$$

If $\mathrm{card}(\Lambda^*) = \mathrm{card}(\prod_{i \in K} \mu_{\xi_i^*})$ we can define I^{α^*} to be K. Otherwise, our assumption on the cardinal arithmetic gives

$$\mathrm{card}\big(\prod_{i \in K} \mu_{\xi_i^*}\big) = \big(\bigcup_{i \in K} \mu_{\xi_i^*}\big)^+ > \mathrm{card}(\Lambda^*).$$

By the choice of α^*, $\mathrm{card}(\Lambda^*) \ge \sup S(\tilde{R}) \ge \bigcup_{i \in K} \mu_{\xi_i^*}$. Hence $\mathrm{card}(\Lambda^*) = \bigcup_{i \in K} \mu_{\xi_i^*}$ and again we can choose I^{α^*} to be K.

It follows, that when $\mathrm{card}(J^*) < \kappa$ we can find I^{α^*} satisfying (6) on the preceding page.

4.4 Case 2: The parameter depends on κ-many coordinates

Remark. If $\bar{\mu}$ is such that each μ_ξ is κ^+ or λ, we have so far proved that θ must be either $\leq \kappa^+$ or $\theta = \lambda$.

For the rest of the proof we assume that the set J^*, given in (8) on the page before, has cardinality κ and for every $K \in [J^*]^{<\kappa}$, $\operatorname{card}\left(\prod_{i \in K} \mu_{\xi_i^*}\right) < \operatorname{card}(\Lambda^*)$. So $\chi^* \leq \operatorname{card}(\Lambda^*)$ holds, where

$$\chi^* = \bigcup_{K \in [J^*]^{<\kappa}} \operatorname{card}\left(\prod_{i \in K} \mu_{\xi_i^*}\right).$$

As we already know that the inequality $\operatorname{card}(\Lambda^*) \leq \operatorname{card}\left(\prod_{i \in J^*} \mu_{\xi_i^*}\right)$ holds, the remaining problem is that why is θ a product of strictly less than κ-many cardinals in $\{\mu_{\xi_i^*} \mid i \in J^*\}$?

Definition 4.4 *Define $\mathcal{E}^*$ to be the set of all sequences $\bar{\varepsilon} = \langle \varepsilon_i \mid i < \kappa \rangle$ such that*

> *for each $i \in J^*$, $\varepsilon_i \in \mu_{\xi_i^*} \smallsetminus \Delta_{\xi_i^*}^{\tilde{R}}$,*
>
> *for each $i \in \kappa \smallsetminus J^*$, $\varepsilon_i = \delta_i^*$, and*
>
> *for every $i < j < \kappa$, $\langle \xi_i^*, \varepsilon_i \rangle \neq \langle \xi_j^*, \varepsilon_j \rangle$.*

Again, to simplify our notation, we write $\pi(\bar{\delta})$ for the sequence $\langle \pi_{\xi_i^*}(\delta_i) \mid i < \kappa \rangle$ when $\bar{\delta}$ is in $\mathcal{E}^*$ and π in $\operatorname{Mps}(\bar{z})$ satisfies that $\{\langle \xi_i^*, \delta_i \rangle \mid i < \kappa\} \subseteq \operatorname{dom}(\pi)$,

Every sequence $\bar{\varepsilon}$ in $\mathcal{E}^*$ determines a $\mathbb{P}(\bar{z})$-name $\tau_{\bar{\varepsilon}}$ for a function from κ into 2. Namely, we define $\tau_{\bar{\varepsilon}}$ to be the name $\pi(\sigma_{\alpha^*})$ where π is the mapping in $\operatorname{Mps}(\bar{z})$ satisfying that $\operatorname{dom}(\pi) = \{\langle \xi_i^*, \delta_i^* \rangle \mid i < \kappa\}$ and $\pi(\bar{\delta}^*) = \bar{\varepsilon}$.

A pair $\langle \bar{\delta}, \bar{\varepsilon} \rangle$ of sequences in $\mathcal{E}^*$ is called a neat pair if for all $i < j < \kappa$, $\langle \xi_i^*, \delta_i \rangle \neq \langle \xi_j^*, \varepsilon_j \rangle$.

Denote the set $\{i \in J^* \mid \delta_i = \varepsilon_i\}$, for $\bar{\delta}, \bar{\varepsilon} \in \mathcal{E}^*$, by $A(\bar{\delta}, \bar{\varepsilon})$.

The sequence $\bar{\delta}^\alpha$ is in $\mathcal{E}^*$ when $\alpha \in \Lambda^*$. Also $\tau_{\bar{\delta}^\alpha}$ is the name σ_α for every $\alpha \in \Lambda^*$. In fact, $\{\tau_{\bar{\varepsilon}} \mid \bar{\varepsilon} \in \mathcal{E}^*\}$ is the collection of all the $\mathbb{P}(\bar{z})$-names which are "isomorphic" to the fixed representative σ_{α^*}. The reason why we introduced "neat pairs of sequences in $\mathcal{E}^*$" is that those names, determined by sequences in a neat pair, can be "coherently moved" around by automorphisms of $\mathbb{P}(\bar{z})$ as follows.

Lemma 4.5 *Suppose $\bar{\delta}^1, \bar{\delta}^2, \bar{\varepsilon}^1, \bar{\varepsilon}^2 \in \mathcal{E}^*$ are such that both $\langle \bar{\delta}^1, \bar{\varepsilon}^1 \rangle$ and $\langle \bar{\delta}^2, \bar{\varepsilon}^2 \rangle$ are neat, and moreover, $A(\bar{\delta}^1, \bar{\varepsilon}^1) = A(\bar{\delta}^2, \bar{\varepsilon}^2)$ holds. Then there is an automorphism $\hat{\pi}$ of $\mathbb{P}(\bar{z})$ such that $\hat{\pi}(\tilde{R}) = \tilde{R}$, $\hat{\pi}(\tau_{\bar{\delta}^1}) = \tau_{\bar{\delta}^2}$ and $\hat{\pi}(\tau_{\bar{\varepsilon}^1}) = \tau_{\bar{\varepsilon}^2}$. Hence for every $p \in \mathbb{P}(\bar{z})$,*

$$p \Vdash_{\mathbb{P}(\bar{z})} \tau_{\bar{\delta}^1} \sim_{\phi, \tilde{R}} \tau_{\bar{\varepsilon}^1} \text{ iff } \hat{\pi}(p) \Vdash_{\mathbb{P}(\bar{z})} \tau_{\bar{\delta}^2} \sim_{\phi, \tilde{R}} \tau_{\bar{\varepsilon}^2}.$$

Proof. There is a mapping π in $\text{Mps}(\bar{z})$ such that $\pi(\bar{\delta}^1) = \bar{\delta}^2$ and $\pi(\bar{\varepsilon}^1) = \bar{\varepsilon}^2$, because the sequences in $\mathcal{E}^*$ are without repetition, both of the pairs are neat, and the equation $A(\bar{\delta}^1, \bar{\varepsilon}^1) = A(\bar{\delta}^2, \bar{\varepsilon}^2)$ holds. Furthermore, π can be chosen so that $\pi{\restriction}\Delta^{\tilde{R}}$ is identity and each $\pi_{\xi_i^*}$ is a permutation of $z_{\xi_i^*}$. Hence π determines an automorphism as wanted. ∎

For technical reasons we define

$$\mathcal{A}^* = \Big\{ I \subseteq \kappa \mid \quad \text{there are } \alpha \neq \beta \in \Lambda^* \text{ such that}$$
$$\langle \bar{\delta}^\alpha, \bar{\delta}^\beta \rangle \text{ is neat and } I \subseteq A(\bar{\delta}^\alpha, \bar{\delta}^\beta) \Big\}.$$

The next lemma explains why we closed the set $\mathcal{A}^*$ under subsets: all the names σ_α, $\alpha \in \Lambda^*$, are forced to be nonequivalent, and moreover, all those names are forced to be nonequivalent, which are determined by a neat pair of sequences agreeing in a smaller set than some pair of the fixed sequences $\bar{\delta}^\alpha$, $\alpha \in \Lambda^*$.

Lemma 4.6 *For all $\bar{\delta}, \bar{\varepsilon} \in \mathcal{E}^*$, if $\langle \bar{\delta}, \bar{\varepsilon} \rangle$ is neat and $A(\bar{\delta}, \bar{\varepsilon})$ is in $\mathcal{A}^*$, then $\mathbf{1} \Vdash_{\mathbb{P}(\bar{z})} \tau_{\bar{\delta}} \not\sim_{\phi, \tilde{R}} \tau_{\bar{\varepsilon}}$.*

Proof. First we fix $\alpha \neq \beta \in \Lambda^*$ and I such that $\langle \bar{\delta}^\alpha, \bar{\delta}^\beta \rangle$ is neat and $I = A(\bar{\delta}, \bar{\varepsilon}) \subseteq A(\bar{\delta}^\alpha, \bar{\delta}^\beta)$. Let $\bar{\delta}'$ be a sequence in $\mathcal{E}^*$ which satisfies that $\bar{\delta}'{\restriction}I = \bar{\delta}^\alpha{\restriction}I$ and for all $i \in J^* \smallsetminus I$, $\delta_i' \notin \{\delta_j^\alpha \mid j < \kappa\}$. Then the pair $\langle \bar{\delta}', \bar{\delta}^\alpha \rangle$ is neat and $A(\bar{\delta}', \bar{\delta}^\alpha) = I$. We want to show that $\mathbf{1} \Vdash_{\mathbb{P}(\bar{z})} \tau_{\bar{\delta}'} \not\sim_{\phi, \tilde{R}} \tau_{\bar{\delta}^\alpha}$, because then it follows from Lemma 4.5 that $\mathbf{1} \Vdash_{\mathbb{P}(\bar{z})} \tau_{\bar{\delta}} \not\sim_{\phi, \tilde{R}} \tau_{\bar{\varepsilon}}$.

Suppose, contrary to this claim, that $p \in \mathbb{P}(\bar{z})$ satisfies

$$p \Vdash_{\mathbb{P}(\bar{z})} \tau_{\bar{\delta}'} \sim_{\phi, \tilde{R}} \tau_{\bar{\delta}^\alpha}.$$

Let J denote the set $A(\bar{\delta}^\alpha, \bar{\delta}^\beta)$ and choose a sequence $\bar{\varepsilon}'$ from $\mathcal{E}^*$ so that $\bar{\delta}^\beta{\restriction}J = \bar{\varepsilon}'{\restriction}J$ and for all $i \in J^* \smallsetminus J$,

$$\varepsilon_i' \notin \Delta_{\xi_i^*}^p \cup \{\delta_j' \mid j < \kappa\} \cup \{\delta_j^\alpha \mid j < \kappa\}.$$

Then the pair $\langle \bar{\varepsilon}', \bar{\delta}^\alpha \rangle$ is neat and $A(\bar{\varepsilon}', \bar{\delta}^\alpha) = J$. By the choice of the names σ_α and σ_β, i.e., by (3) on page 19, $\mathbf{1} \Vdash_{\mathbb{P}(\bar{z})} \sigma_\alpha \not\sim_{\phi,R} \sigma_\beta$. Once more, it follows from Lemma 4.5 that

$$\mathbf{1} \Vdash_{\mathbb{P}(\bar{z})} \sigma_\alpha = \tau_{\bar{\delta}^\alpha} \not\sim_{\phi,R} \tau_{\bar{\varepsilon}'}.$$

Choose π from $\mathrm{Mps}(\bar{z})$ so that $\pi(\tilde{R}) = \tilde{R}$, $\pi(\bar{\delta}') = \bar{\delta}'$, $\pi(\bar{\delta}^\alpha) = \bar{\varepsilon}'$, $\pi{\restriction}(\pi[\Delta^p] \cap \Delta^p)$ is identity, and π determines an automorphism $\hat{\pi}$ of $\mathbb{P}(\bar{z})$. This is possible by the choice of the sequence $\bar{\varepsilon}'$. Since $A(\bar{\delta}', \bar{\varepsilon}') = A(\bar{\delta}', \bar{\delta}^\alpha)$ and $\langle \bar{\delta}', \bar{\varepsilon}' \rangle$ is a neat pair, it follows from Lemma 4.5 that

$$\hat{\pi}(p) \Vdash_{\mathbb{P}(\bar{z})} \tau_{\bar{\delta}'} \sim_{\phi,\tilde{R}} \tau_{\bar{\varepsilon}'}.$$

Now there is $q \in \mathbb{P}(\bar{z})$ satisfying $q \leq p$ and $q \leq \hat{\pi}(p)$. Since $\sim_{\phi,\tilde{R}}$ is a name for an equivalence relation, $q \Vdash_{\mathbb{P}(\bar{z})} \tau_{\bar{\delta}^\alpha} \sim_{\phi,\tilde{R}} \tau_{\bar{\varepsilon}'}$, a contradiction. $\blacksquare$4.6

Next we want to show that there is always a small set of indices outside of $\mathcal{A}^*$.

Lemma 4.7 *When J^* has cardinality κ there are $p \in \mathbb{P}(\bar{z})$ and a neat pair $\langle \bar{\delta}, \bar{\varepsilon} \rangle$ of sequences in $\mathcal{E}^*$ satisfying that*

$$A(\bar{\delta}, \bar{\varepsilon}) \in [J^*]^{<\kappa} \text{ and } p \Vdash_{\mathbb{P}(\bar{z})} \tau_{\bar{\delta}} \sim_{\phi,\tilde{R}} \tau_{\bar{\varepsilon}}.$$

Proof. First of all, for every $i \in J^*$ and $\eta \in {}^i 2$ we fix an ordinal β_η from $\mu_{\xi_i^*} \smallsetminus \Delta_{\xi_i^*}^{\tilde{R}}$ so that for all $i, j \in J^*$, $\eta \in {}^i 2$, and $\nu \in {}^j 2$, $\beta_\eta = \beta_\nu$ iff $i = j$ and $\eta = \nu$. Fix also a coordinate $\zeta < \lambda$ so that $\mu_\zeta > \theta$ and $\zeta \notin Z$ (ζ is outside of $\mathbb{P}(\bar{z})$). Suppose G is a P_{μ_ζ}-generic set over V. For any function $u \in ({}^\kappa 2)^{V[G]}$, let $\bar{\delta}^u$ denote the following sequence: $\bar{\delta}^u = \langle \delta_i^u \mid i < \kappa \rangle$, $\delta_i^u = \beta_{u{\restriction}i}$ if $i \in J^*$, and $\delta_i^u = \delta_i^*$ otherwise. Then each of the sequences $\bar{\delta}^u$ is in $(\mathcal{E}^*)^{V[G]}$. Moreover, $\langle \bar{\delta}^u, \bar{\delta}^v \rangle$ is a neat pair of for all u and v in $({}^\kappa 2)^{V[G]}$.

Let H be a $\mathbb{P}(\bar{z})$-generic set over $V[G]$. In $V[G]$, there are at least μ_ζ many different functions from κ into 2. By the assumption (2) on page 15 and Conclusion 3.6, there are only θ-many equivalence classes of $\sim_{\phi,R}$ in $V[G][H]$. It follows, that for some $p \in H$ and $u \neq v \in ({}^\kappa 2)^{V[G]}$ the following holds in $V[G]$,

$$p \Vdash_{\mathbb{P}(\bar{z})} \tau_{\bar{\delta}^u} \sim_{\phi,\tilde{R}} \tau_{\bar{\delta}^v}.$$

By the definition of the ordinals β_ν, we have that $A(\bar{\delta}^u, \bar{\delta}^v) = \{i \in J^* \mid u{\restriction}i = v{\restriction}i\} \in [J^*]^{<\kappa}$, and hence $A(\bar{\delta}^u, \bar{\delta}^v)$ is in V.

24

Now, in V, we can fix a neat pair $\langle \bar{\varepsilon}^1, \bar{\varepsilon}^2 \rangle$ of sequences in $\mathcal{E}^*$ such that $A(\bar{\varepsilon}^1, \bar{\varepsilon}^2) = A(\bar{\delta}^u, \bar{\delta}^v)$. Let $\pi \in (\mathrm{Mps}(\bar{z}))^{V[G]}$ be such that it determines, in $V[G]$, an automorphism $\hat{\pi}$ of $\mathbb{P}(\bar{z})$ satisfying $\hat{\pi}(\tilde{R}) = \tilde{R}$, $\hat{\pi}(\bar{\delta}^u) = \bar{\varepsilon}^1$, and $\hat{\pi}(\bar{\delta}^v) = \bar{\varepsilon}^2$. For such π in $V[G]$, we have that $\hat{\pi}(p) \Vdash_{\mathbb{P}(\bar{z})} \tau_{\bar{\varepsilon}^1} \sim_{\phi, \tilde{R}} \tau_{\bar{\varepsilon}^2}$. Note, that the condition $q = \hat{\pi}(p)$ is in V. From the equivalence of the forcings $P_{\mu_\zeta} \times \mathbb{P}(\bar{z})$ and $\mathbb{P}(\bar{z}) \times P_{\mu_\zeta}$, together with Lemma 3.5(c), it follows that already in V,

$$q \Vdash_{\mathbb{P}(\bar{z})} \tau_{\bar{\varepsilon}^1} \sim_{\phi, \tilde{R}} \tau_{\bar{\varepsilon}^2}.$$

<div align="right">■4.7</div>

Finally, we claim that $\mathrm{card}(\Lambda^*) = \chi^*$, and thus we can satisfy (7) on page 20. Suppose, contrary to this claim, that $\mathrm{card}(\Lambda^*) > \chi^*$. In the lemma below, we show that then all the subsets of J^* of cardinality $< \kappa$ are in $\mathcal{A}^*$. It follows from Lemma 4.6, that for all $\bar{\delta}, \bar{\varepsilon} \in \mathcal{E}^*$, if $\langle \bar{\delta}, \bar{\varepsilon} \rangle$ is neat and $A(\bar{\delta}, \bar{\varepsilon})$ is of cardinality $< \kappa$, then $\mathbf{1} \Vdash_{\mathbb{P}(\bar{z})} \tau_{\bar{\delta}} \not\sim_{\phi, \tilde{R}} \tau_{\bar{\varepsilon}}$. By Lemma 4.7, this leads to a contradiction. So it remains to prove the following last lemma.

Lemma 4.8 If $\mathrm{card}(\Lambda^*) > \chi^*$ then $[J^*]^{<\kappa} \subseteq \mathcal{A}^*$.

Proof. Fix a set K from $[J^*]^{<\kappa}$. Since

$$\mathrm{card}(\Lambda^*) > \chi^* \geq \mathrm{card}\Big(\prod_{i \in K} \mu_{\xi_i^*} \Big) \geq 2^\kappa,$$

there is $X_1 \subseteq \Lambda^*$ of cardinality $(2^\kappa)^+$ such that for all $\alpha \neq \beta \in X_1$, $K \subseteq A(\bar{\delta}^\alpha, \bar{\delta}^\beta)$. By Δ-lemma one can find $X_2 \in [X_1]^{(2^\kappa)^+}$ such that for all $\alpha \neq \beta \in X_2$, the intersection $\{\delta_i^\alpha \mid i < \kappa\} \cap \{\delta_i^\beta \mid i < \kappa\}$ is some fixed set Ξ. There are also $I \subseteq \kappa$ and $X_3 \in [X_2]^{(2^\kappa)^+}$ such that for all $\alpha \in X_3$, $\{i < \kappa \mid \delta_i^\alpha \in \Xi\} = I$. Hence there is $\alpha \neq \beta \in X_3$ with $\bar{\delta}^\alpha {\restriction} I = \bar{\delta}^\beta {\restriction} I$ and $\{\delta_i^\alpha \mid i \in \kappa \smallsetminus I\} \cap \{\delta_i^\beta \mid i \in \kappa \smallsetminus I\} = \emptyset$, i.e., $\langle \bar{\delta}^\alpha, \bar{\delta}^\beta \rangle$ forms a neat pair with $K \subseteq I = A(\bar{\delta}^\alpha, \bar{\delta}^\beta)$. ■

5 Remarks

In fact, we needed the assumption that $\sim_{\phi, R}$ is Σ_1^1-definable over $H(\kappa)$ only in the proof of the absoluteness of $\sim_{\phi, R}$ for extensions over the subforcing $\mathbb{P}(\bar{z})$ and the whole forcing $\mathbb{P}(\bar{\mu})$, i.e., in the proof of Lemma 3.5. From Conclusion 3.6 it follows that Theorem 1 holds for all equivalence relations

<div align="center">25</div>

which are definable over $H(\kappa)$ using a parameter and a sentence which is a Boolean combination of a sentence containing one second order existential quantifier (Σ_1^1-sentence) and a sentence containing one second order universal quantifier (Π_1^1-sentence). This observation has a minor application in [SV]. Note, that there is in preparation by Shelah a continuation of this paper where the result is generalized (for example the singular cardinal hypothesis will be eliminated). For a more general treatment of the subject see [She].

Note that the possible numbers of κ-branches in trees of cardinality κ and the possible numbers of equivalence classes of Σ_1^1-equivalence relations are consistent wise almost the same. The main difference is of course the number κ^+ (and 0, too). Particularly, if $\bar{\chi} = \langle \chi_i \mid i < \gamma \rangle$ is a sequence of nonzero cardinals such that $\gamma \leq \kappa$ and for every $i < \gamma$, there exists a tree T_i with $\mathrm{card}(T_i) \leq \kappa$ and $\mathrm{card}(\mathrm{Br}_\kappa(T_i)) = \chi_i$, then there exists a tree T with $\mathrm{card}(T) \leq \kappa$ and $\mathrm{card}(\mathrm{Br}_\kappa(T)) = \bigcup_{i<\gamma} \chi_i$, and furthermore, there exists a tree T with $\mathrm{card}(T) \leq \kappa$ and $\mathrm{card}(\mathrm{Br}_\kappa(T)) = \mathrm{card}(\prod_{i<\gamma} \chi_i)$, provided that $\gamma < \kappa$ and $\kappa^{<\kappa} = \kappa$. Therefore, a variant of the Theorem 1 holds, where instead of the possible numbers of equivalence classes one considers the numbers of κ-branches through trees of cardinality κ.

The following facts are also useful to know, when applying the theorem proved. Write $\mathrm{Fn}(\kappa, 2, \kappa)$ for the ordinary Cohen-forcing which adds a generic subset of κ, i.e., the forcing

$$\{\eta \mid \eta \text{ is a partial function from } \kappa \text{ into } 2 \text{ and } \mathrm{card}(\eta) < \kappa\}$$

ordered by reverse inclusion.

Fact 5.1

a) *There is a dense subset $Q \subseteq \mathrm{Fn}(\kappa, 2, \kappa)$ and a dense embedding of Q into P_κ (where P_κ is the forcing adding a tree with κ-many branches, see Definition 2.1).*

b) *Every subforcing $\mathbb{P}(\bar{z})$ of $\mathbb{P}(\bar{\mu})$ is equivalent to $\mathrm{Fn}(\kappa, 2, \kappa)$ provided that the length of $\bar{z}$ is at most κ and each z_ξ has cardinality κ.*

c) *The forcing $\mathbb{P}(\bar{\mu})$ is locally κ Cohen, i.e., every subset Q of $\mathbb{P}(\bar{\mu})$ of size $\leq \kappa$ is included in a complete subforcing Q' of $\mathbb{P}(\bar{\mu})$ so that Q' is equivalent to $\mathrm{Fn}(\kappa, 2, \kappa)$.*

d) *Assume that κ is a weakly compact cardinal, and V is such that κ remains weakly compact after forcing with $\mathrm{Fn}(\kappa, 2, \kappa)$. Then every locally κ Cohen forcing preserves weakly compactness of κ.*

Note in the last claim, that if κ is a weakly compact cardinal then, using upward Easton forcing, it is possible to have a generic extension $V[H]$ such that κ is weakly compact in $V[H]$ and κ remains weakly compact in all extensions $V[H][G]$, where G is $\mathrm{Fn}(\kappa, 2, \kappa)$-generic over $V[H]$ (Silver). These facts are applied in [SV].

References

[Jec71] Thomas J. Jech. Trees. *J. Symbolic Logic*, 36:1–14, 1971.

[Jec97] Thomas Jech. *Set theory*. Springer-Verlag, Berlin, second edition, 1997.

[She] Saharon Shelah. Strong dichotomy of cardinality. *Results in Mathematics*, to appear.

[SV] Saharon Shelah and Pauli Väisänen. The number of $L_{\infty\kappa}$-equivalent non-isomorphic models for κ weakly compact. No. 718 in the list of Shelah's publications.

Saharon Shelah:

Institute of Mathematics
The Hebrew University
Jerusalem. Israel

Rutgers University
Hill Ctr-Busch
New Brunswick. New Jersey 08903
shelah@math.rutgers.edu

Pauli Väisänen:

Department of Mathematics
P.O. Box 4
00014 University of Helsinki
Finland
pauli.vaisanen@helsinki.fi